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Final Exam , Math 531, Spring 2014

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QUESTION 1. 1)Let R be a finite integral domain. Prove that R is a field.

2) Let R be a finite commutative ring with $1 \neq 0$, and let P be a prime ideal of of R. Prove that P is a maximal ideal of R.

3) Let R be a commutative ring with $1 \neq 0$, and I be an ideal of R. Prove that $I/(M \cap I)$ is a field for every maximal ideal M of R where $I \not\subseteq M$.

QUESTION 2. 1) Let $D = \{f(x) \in Q[x] \mid f(\sqrt[3]{2}) = 0\}$. Prove that D is a maximal ideal of $Q[X]$

2) Let R be a finite commutative ring with $1 \neq 0$ such that $|R| = p$ for some prime integer p. Prove that R is ring-isomorphic to Z_p.

3) Prove that $Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$ is not ring-isomorphic to $Z[\sqrt{5}\} = \{c + d\sqrt{5} \mid c, d \in Z\}$.

QUESTION 3. 1) Let R, S be commutative rings with $1 \neq 0$, and $F : R \to S$ be a ring homomorphism such that $F(1) \neq 0_s$ and $F(1) \neq 1_S$. Prove that S is not an integral domain.

2) Let $F \subset K$ be field extensions, $d(x), g(x)$ be monic irreducible polynomials over F such that $d(a) = g(b) = 0$ for some $a, b \in K$. Given $deg(d) = 8$ and $deg(g) = 15$. Prove that $g(x)$ is irreducible over $F(a)$.

QUESTION 4. 1) Let $R = Z \times Z$, $S = \{(a, 0)|a \neq 0$ and $a \in Z\}$. Prove that R_S is a field.

2) Let $R = Z \times Z$, $S = \{(a, b)|a \notin 3Z$ and $b \notin 5Z\}$. First show that S is a multiplicatively closed set of R. How many distinct prime ideals does R_S have ? explain.

3) Let $I = 12Z$ is an ideal of Z. Describe all units in $R = Z[x]/I[x]$, i.e., $U(R) = \{f(x) + I[x] \in R$ find conditions on the coefficients of $f(x)\}$.

QUESTION 5. Let a be a positive integer such that $gcd(a, 2) = 1$, and $f(x) = x^4 + 2x^2 + 2a \in Q[x]$.

(i) Prove that $f(x)$ is irreducible over Q.

(ii) Let b be a root of $f(x)$. Find a basis for $[Q(b^2) : Q]$. In view of $f(x)$, find the unique monic irreducible polynomial $L(x) \in Q[x]$ such that $L(b^2) = 0$.

(iii) Prove that $Q(b^3) = Q(b)$.

(iv) Let $S = \{s_1, ..., s_n\}$ be a basis for $Q(b)$ over Q. Since $b^{-1} \in Q(b)$, then we know that b^{-1} can be written uniquely as $c_1 s_1 + ... + c_n s_n$ where the c_i's are in Q. Find $c_1, c_2, ..., c_n$ (you may write each c_i in terms of a).

Faculty information

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